JOURNAL OF COMPUTATIONAL PHYSICS **41**, 245–255 (1981)

The Asymptotic Solution of the Orr–Sommerfeld Equation at Large Distance from a Shear Layer

PETER A. MONKEWITZ*

Department of Aerospace Engineering, University of Southern California, Los Angeles, California 90007

AND

MARTIN A. MONKEWITZ†

Institut für Aerodynamik, Eidgenössische Technische Hochschule (ETH), Zürich, Switzerland

Received March 7, 1980

The two-dimensional Orr–Sommerfeld equation in a domain of infinite cross-stream extent is commonly solved by joining a numerical computation at some finite distance from the shear layer to the "free-stream" normal mode solutions for the mean velocity $U = \text{const}$. In this paper these solutions are improved by introducing the asymptotic expansions for U and U_{yy} into the O–S equation. The resulting inhomogeneous "free-stream" O–S equation yields the asymptotic corrections to the $U = \text{const}$ solutions analytically. The Blasius boundary layer, the Blasius mixing layer and also the $\tanh(y)$-profile are considered as mean velocity profiles. The results are exploited to correct the starting data of numerical computations for the finite computational domain which caused problems in previous studies aimed at the calculation of higher modes. It is shown that, if the starting data are given by an N-term asymptotic expansion (usually two terms in this paper as opposed to the commonly used leading term only), an eigenvalue approaches its asymptotic value at the same rate as the mean velocity approaches its N-term asymptotic expansion when the computational domain is extended further and further away from the shear layer. With the considered mean velocity profiles which have asymptotic expansions in terms of powers of exponentials, this leads to a drastic increase in significant figures of an eigenvalue for a given domain.

1. INTRODUCTION

In recent work aimed at the clarification of the full spectrum of the O–S equation in an infinite domain, difficulties have been reported which were traced back to the necessarily finite computational domain. Mack [5], in his numerical study of the

* Present address: Mechanics and Structures Department, UCLA, Los Angeles, Calif. 90024.
† Present address: Flugzeugwerk Emmen, Emmen, Switzerland.

0021-9991/81/060245–11$02.00/0

temporal eigenvalue spectrum of the Blasius boundary layer, reports that he has to start the numerical integration for the higher modes very far from the wall. For instance, at a Reynolds-number of 1000 he has to start as $y_s = 14$ to obtain the highest discrete mode 10. This implies a very refined computation technique as the deviation of the mean velocity from its asymptotic value 1 is important all the way out to $y_s = 14$, where its value is of the order of 10^{-18}. Also Antar and Benek [1] show some results where higher mode eigenvalues depend strongly on the starting point y_s of their numerical integration.

In this paper a method is divised to improve this situation by correcting the usual "free-stream" starting data (obtained for constant mean velocity) for the finite computational domain. This is achieved by using the asymptotic expansion of the mean velocity profile and constructing the corresponding asymptotic expansion of the solution of the O–S equation. Equivalently this amounts to an approximate analytic integration from infinity to the starting point y_s of the numerical calculation. Independently a similar approach has been adopted by Ng and Reid [8] in the limiting case of large Reynolds number. In the following the asymptotic behaviour of different Blasius layers (boundary layer, free shear layer) is reviewed in Section 2 and used in Section 3 to develop the asymptotic behaviour of the solutions of the O–S equation.

2. The Asymptotics of the Blasius Equation

We consider two cases of two-dimensional shear layers sketched in Fig. 1: the laminar boundary layer over a flat plate, denoted subsequently by (BL), and the laminar mixing layer, denoted by (ML). The governing boundary-layer equations (see, e.g., Schlichting, [9]) accept a similarity solution which was first obtained by Blasius [2]. With asterisks denoting dimensional quantities throughout the paper, the similarity variable

$$y = \left(\frac{U_\infty^*}{\nu^* x^*}\right)^{1/2} y^* \tag{1}$$

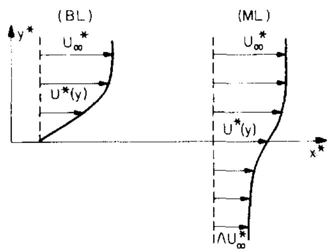

Fig. 1. The shear layers under consideration.

is introduced, where the reference velocity U^*_∞ is taken as the constant free-stream velocity at large positive y^* and where ν^* denotes the kinematic viscosity. With this similarity variable the boundary-layer equations reduce to the well-known Blasius equation

$$2D^3F(y) + F(y)\,D^2F(y) = 0 \qquad \text{with} \quad D^m \equiv d^m/dy^m. \tag{2}$$

The nondimensional velocity components U and V in the x^* and y^* direction are obtained through

$$\begin{aligned} U &= U^*/U^*_\infty = DF(y), \\ V &= V^*/U^*_\infty = \tfrac{1}{2}R_x^{-1/2}[\,yDF(y) - F(y)] \qquad \text{with} \quad R_x = U^*_\infty x^*/\nu^*. \end{aligned} \tag{3}$$

The boundary conditions for F in the two cases (BL) and (ML) are given by

$$\begin{array}{rr} \text{(BL):} & \text{(ML):} \\ DF(+\infty) = 1, & DF(+\infty) = 1, \\ F(0) = 0, & F(0) = 0, \\ DF(0) = 0, & DF(-\infty) = \Lambda. \end{array} \tag{4}$$

It has to be noted that in the case (ML) the condition $F(0) = 0$, which fixes $V = 0$ at the location $y = 0$, is arbitrary. But as Lock [4] and others have pointed out this arbitrariness amounts only to a shift of the velocity profile along the y-axis or equivalently to the free choice of V at $y = +\infty$.

In the remainder of this section the asymptotic behaviour of $F(y)$ for large $|y|$ is discussed. The presented material is mostly a review of the work by Blasius [2] and Lock [4] with some extensions and generalizations. To cover both cases (BL) and (ML) the following general condition for $F(y)$ at infinity is considered:

$$DF(s\infty) = \lambda \qquad \text{with} \quad s = \pm 1. \tag{5}$$

First, we consider $\lambda > 0$. In this case the leading asymptotic term of F has to be a linear function f_0 of y. If we insert the formal expansion

$$F_{\mathrm{as}} = \sum_{n=0}^{\infty} f_n \tag{6}$$

into the Blasius equation (2) and order the terms we obtain the set of equations

$$2D^3 f_1 + f_0 D^2 f_1 = 0, \tag{7a}$$

$$2D^3 f_n + f_0 D^2 f_n = -\sum_{k=1}^{n-1} D^2 f_k f_{n-k} \qquad (n > 1) \tag{7b}$$

with $f_0 = 2\sqrt{\lambda}\,z$, $z \equiv \sqrt{\lambda}\,(y - \delta)/2$, and boundary conditions $f_n(s\infty) = 0$ $(n \geqslant 1)$.

In this formulation all integration constants arising from the integration of the Df_n have been gathered into δ so that, considering $Df_0 = \lambda$, the boundary condition (5) translates into the above condition for the f_n. The solution of the first equation (7a) is readily obtained as $D^2 f_1 = \gamma \exp(-z^2)$ with a second free constant γ. The particular solutions of the second order equation have been given in integral form by Blasius [2]. In the following, the results for Df_n are presented up to $n = 3$, which is the highest n accessible to closed form integration.

$$U_{\text{as}} = \sum_{n=0}^{\infty} Df_n; \qquad y \to s\infty \ (s = \pm 1),\ \lambda > 0, \tag{8}$$

with

$$\begin{aligned}
Df_0 &= \lambda, \\
Df_1 &= -\gamma \lambda^{-1/2} \pi^{1/2} s \operatorname{erfc}(sz), \\
Df_2 &= \gamma^2 \lambda^{-2} \{-(\pi/2) \operatorname{erfc}^2(sz) - \pi^{1/2} z s e^{-z^2} \operatorname{erfc}(sz) + e^{-2z^2}\}, \\
Df_3 &= \gamma^3 \lambda^{-7/2} \{-\tfrac{1}{2} \pi^{3/2} s \operatorname{erfc}^3(sz) - (\pi/4)\, z(2z^2 + 7)\, e^{-z^2} \operatorname{erfc}^2(sz) \\
&\quad + \pi^{1/2}(z^2 + 3)\, s e^{-2z^2} \operatorname{erfc}(sz) - 2^{3/2} \pi^{1/2} s e^{-z^2} \operatorname{erfc}(2^{1/2} sz) \\
&\quad + \tfrac{1}{4} 3^{3/2} \pi^{1/2} s \operatorname{erfc}(3^{1/2} sz) - \tfrac{1}{2} z e^{-3z^2}\},
\end{aligned}$$

with the definition

$$\operatorname{erfc}(z) = 2\pi^{-1/2} \int_z^{\infty} e^{-t^2}\, dt; \qquad \frac{d}{dz} \operatorname{erfc}(z) = -2\pi^{-1/2} e^{-z^2}.$$

The leading asymptotic term of all the expressions, $D^m f_n$, which is useful for quick evaluations, is obtained directly from the system of equations (7) as

$$D^m f_n \propto \lambda^{1/2} \left(\frac{\gamma}{\lambda^{3/2}}\right)^n \frac{a_n (-\lambda^{1/2} n z)^m}{n^2 z^{3n-1}} e^{-nz^2} \{1 + O(z^{-2})\}, \qquad n \geqslant 1;\ m = 0, 1,..., \tag{9a}$$

with the recurrence relation for the a_n

$$\begin{aligned}
a_1 &= 1, \\
a_{n+1} &= \frac{1}{2n} \sum_{k=1}^{n} \frac{a_k a_{n+1-k}}{(n+1-k)^2}.
\end{aligned} \tag{9b}$$

These approximations are also readily verified by replacing the function $\operatorname{erfc}(z)$ in expressions (8) by its asymptotic expansion.

The two free constants γ, in (8), and δ, which appears in the definition of the variable z, remain to be determined from a comparison with a numerical calculation F_{num}. δ is obtained as $\delta = \lim_{y \to s\infty} (y - \lambda^{-1} F_{\text{num}})$ and γ is determined by matching, say, U_{as} given by (8) to DF_{num} at a suitable y.

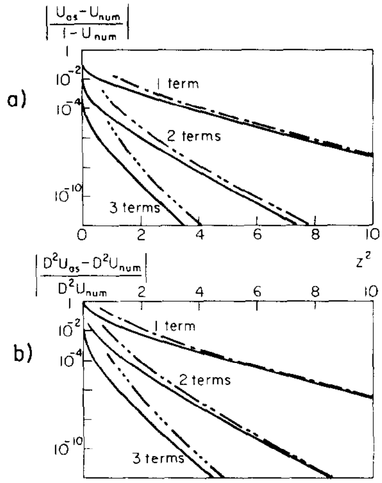

FIG 2. (a) Relative error of the asymptotic expansion $(1-U_{as})$ with asymptotic leading terms: $-\cdot-$, $\gamma(1/4z^3)\exp(-z^2)$; $-\cdot\cdot-$, $\gamma^2(5/96z^6)\exp(-2z^2)$; $-\cdots-$, $\gamma^3(17/1728z^9)\exp(-3z^2)$. (b) Corresponding relative error of D^2U_{as} with asymptotic leading terms: $-\cdot-$, $\gamma(1/z^3)\exp(-z^2)$; $-\cdot\cdot-$, $\gamma^2(15/32z^6)\exp(-2z^2)$; $-\cdots-$, $\gamma^3(17/108z^9)\exp(-3z^2)$.

For the *boundary layer* with $\lambda=1$ and $s=+1$ this comparison has been made with a calculation to 14 significant digits by M. Monkewitz yielding $\delta=1.7207876575207$. γ was then obtained by matching $\sum_{n=0}^{3} Df_n$ given by (8) to DF_{num} at $y=5.65$ yielding the value $\gamma=0.233727621285$ (cf. also Mack [5], who used only one term to determine γ). For subsequent reference, the relative errors of U_{as} and D^2U_{as} are plotted in Fig. 2. The asymptotic leading terms for these errors which are also plotted, are taken from (9).

For the *mixing layer*, γ and δ have been determined in two cases by comparison to calculations of Lock [4] and are listed in Table I.

To conclude this section, we also review the special case of the mixing layer with $\Lambda=0$ although the results are questionable because the boundary-layer assumption $V \ll U$ is clearly violated on the zero U-velocity side. Considering the boundary conditions (4) with $DF(-\infty)=0$, $F(-\infty)$ has to be a negative constant σ. So the following asymptotic expansion for F has been given by Lock [4]:

$$F_{as}=\sigma\left\{1+\sum_{n=1}^{\infty} b_n\left(\frac{\tau}{\sigma}e^{-(\sigma/2)y}\right)^n\right\}, \qquad y\to-\infty;\quad \sigma<0. \tag{10a}$$

Inserting this expansion into the Blasius equation (2) yields a recurrence relation for the coefficients b_n,

$$b_1=1,\qquad b_{n+1}=\frac{1}{n(n+1)^2}\sum_{k=1}^{n} k^2 b_k b_{n+1-k}. \tag{10b}$$

TABLE I

Asymptotic Expansions for a Blasius Mixing Layer Profile with Boundary Conditions (4)

	As. expansion for $y \to +\infty$ $s=+1$	As. expansion for $y \to -\infty$ $s=-1$	Source of numerical calculation
$\Lambda=0$	(8) with $\lambda=1$ $\delta=0.529$ $\gamma=0.165$	(10) with $\sigma=-1.239$ $\tau=1.63$	Lock [4, Table VI]
$\Lambda=0.501$	(8) with $\lambda=1$ $\delta=0.283$ $\gamma=0.109$	(8) with $\lambda=0.501$ $\delta=0.749$ $\gamma=0.162$	Lock [4, Table VII]

Again, the two free constants σ and τ are obtained from a comparison with a numerical calculation by Lock [4]; they are also listed in Table I.

3. The Asymptotics of the Orr–Sommerfeld Equation for Large $|y|$

For the parallel stability analysis of a Blasius layer at a fixed downstream location x_0^*, we chose the length scale $L^* = (\nu^* x_0^*/U_\infty^*)^{1/2}$ and the velocity scale U_∞^* for nondimensionalisation. Seeking normal mode solutions, we write, for the stream-function $\Phi(x, y, t)$,

$$\Phi(x, y, t) = \phi(y)\, e^{i\alpha(x-ct)}. \tag{11}$$

With this we obtain the O–S equation for $\phi(y)$ in the usual way:

$$\{(D^2-\alpha^2)^2 - i\alpha R\,[(U(y)-c)(D^2-\alpha^2) - D^2U(y)]\}\,\phi = 0 \tag{12}$$

with $R = L^* U_\infty^*/\nu^*$ and $\phi = D\phi = 0$ at the boundaries of the domain.

To investigate the behaviour of $\phi(y)$ at large $|y|$, we introduce the appropriate asymptotic expansion for $U(y)$ and $D^2 U(y)$ into (12) and at the same time substitute the formal expansion $\sum_{n=0}^{\infty} \varphi_n$ for ϕ. With the boundary condition (5) for U, we obtain the following equations by collecting terms of equal order:

$$\{D^4 - (\alpha^2+\beta^2)\, D^2 + \alpha^2\beta^2\}\, \varphi_0 = 0$$

with (13a)

$$\beta^2 = \alpha^2 + i\alpha R(\lambda - c),$$

$$\{D^4 - (\alpha^2+\beta^2)\, D^2 + \alpha^2\beta^2\}\, \varphi_1 = i\alpha R\,\{U_1(D^2-\alpha^2) - D^2 U_1\}\, \varphi_0 \tag{13b}$$

with

$$U(y) \propto \lambda + U_1 + U_2 + \cdots,$$
$$D^2 U(y) \propto 0 + D^2 U_1 + D^2 U_2 + \cdots.$$

The first equation (13a) has the well-known "free-stream" solutions

$$\varphi_0^{(\pm\alpha)} = C^{(\pm\alpha)} e^{\pm\alpha y},$$
$$\varphi_0^{(\pm\beta)} = C^{(\pm\beta)} e^{\pm\beta y}. \tag{14}$$

In the following, we restrict ourselves to the discussion of the first order equation (13b) for the correction to these "free-stream" solutions. For the largest part of this section we will also specify U_1 and $D^2 U_1$ as being given by (8), i.e., by Df_1 and $D^3 f_1$. Thus, the right-hand side of (13b) reads as

$$-\gamma \frac{\beta^2 - \alpha^2}{\lambda - c} \{\pi^{1/2} \lambda^{-1/2} s \operatorname{erfc}(sz)(D^2 - \alpha^2) - \lambda^{1/2} z e^{-z^2}\} \varphi_0$$
$$\text{for} \quad y \to s\infty \, (s = \pm 1); \lambda > 0. \tag{15}$$

The asymptotic behaviour of this expression is given by an algebraic function times $\varphi_0 \exp(-z^2)$, which suggests the following trial solution for φ_1:

$$\varphi_1 = \varphi_0 \Big\{ A e^{-z^2} + sB(z) \operatorname{erfc}(sz)$$
$$+ \sum_n sC_n \exp(2c_n z + c_n^2) \operatorname{erfc}[s(z + c_n)] \Big\}. \tag{16}$$

Insertion of this trial solution into Eq. (13b) yields a cubic equation for c_n and, after a lengthy calculation, the coefficients A, $B(z)$ and the three C_n. The solution is conveniently given separately for the pressure- and vorticity-mode correction:

$$\varphi_1^{(\pm\alpha)} = \varphi_0^{(\pm\alpha)} \frac{\gamma \pi^{1/2}}{\lambda^{1/2}(\lambda - c)}$$
$$\times \Big\{ -\exp[\mp 4\alpha\lambda^{-1/2} z + 4\alpha^2 \lambda^{-1}] \, s \operatorname{erfc}[s(z \mp 2\alpha\lambda^{-1/2})]$$
$$+ \frac{(\mp\alpha + \beta)}{2\beta} \exp[2\lambda^{-1/2}(\mp\alpha + \beta) z + \lambda^{-1}(\mp\alpha + \beta)^2]$$
$$\times s \operatorname{erfc}[s(z + \lambda^{-1/2}(\mp\alpha + \beta))]$$
$$- \frac{(\mp\alpha - \beta)}{2\beta} \exp[2\lambda^{-1/2}(\mp\alpha - \beta) z + \lambda^{-1}(\mp\alpha - \beta)^2]$$
$$\times s \operatorname{erfc}[s(z + \lambda^{-1/2}(\mp\alpha - \beta))] \Big\}, \tag{17a}$$

$$\varphi_1^{(\pm\beta)} = \varphi_0^{(\pm\beta)} \frac{\gamma\pi^{1/2}}{\lambda^{1/2}(\lambda - c)} \left\{ \pm \frac{\beta^2 - \alpha^2}{\beta(\lambda\pi)^{1/2}} \exp[-z^2] \right.$$
$$+ \left[\mp \frac{\beta^2 - \alpha^2}{\beta\lambda^{1/2}} z + \frac{5\beta^2 - \alpha^2}{4\beta^2} \right] s \operatorname{erfc}[sz]$$
$$+ \frac{3\beta^2 + \alpha^2}{4\beta^2} \exp[\mp 4\beta\lambda^{-1/2} z + 4\beta^2\lambda^{-1}]$$
$$\times s \operatorname{erfc}[s(z \mp 2\beta\lambda^{-1/2})]$$
$$- \exp[2\lambda^{-1/2}(\mp\beta + \alpha) z + \lambda^{-1}(\mp\beta + \alpha)^2]$$
$$\times s \operatorname{erfc}[s(z + \lambda^{-1/2}(\mp\beta + \alpha))]$$
$$- \exp[2\lambda^{-1/2}(\mp\beta - \alpha) z + \lambda^{-1}(\mp\beta - \alpha)^2]$$
$$\left. \times s \operatorname{erfc}[s(z + \lambda^{-1/2}(\mp\beta - \alpha))] \right\}. \tag{17b}$$

If the asymptotic expansions for the error functions is introduced into expressions (17) we find that the derivatives of φ_1 show the following behaviour:

$$\left| \frac{D^m \varphi_1^{(\pm\alpha)}}{D^m \varphi_0^{(\pm\alpha)}} \right| = O\left[\left| \frac{\beta}{\alpha} \right|^m \left| \frac{\varphi_1^{(\pm\alpha)}}{\varphi_0^{(\pm\alpha)}} \right| \right],$$
$$\left| \frac{D^m \varphi_1^{(\pm\beta)}}{D^m \varphi_0^{(\pm\beta)}} \right| = O\left[\left| \frac{\varphi_1^{(\pm\beta)}}{\varphi_0^{(\pm\beta)}} \right| \right] \tag{18}$$

This means that the corrections $D^m \varphi_1^{(\pm\alpha)}$ to the higher derivatives $D^m \varphi_0^{(\pm\alpha)}$ of the pressure mode are the most critical.

The results (17) are now easily exploited to improve a numerical calculation: At each cycle of the eigenvalue shooting algorithm the integration is started at some finite y_s with the improved starting data $D^m[\varphi_0^{(-\alpha)} + \varphi_1^{(-\alpha)}](y = y_s)$ and $D^m[\varphi_0^{(-\beta)} + \varphi_1^{(-\beta)}](y = y_s)$ $(m = 0,..., 3)$, Thereby, expressions (17) for φ_1 have to be evaluated with the approximate eigenvalue of the actual iteration cycle. The improvement by this procedure is illustrated with a temporal boundary-layer calculation using a program of M. Monkewitz [7] to obtain the dependence of the eigenvalue c on the starting location $z_s = (y_s - \delta)/2$ of the numerical intergration for the modes 1 (Tollmien–Schlichting mode), 5 and 9 at a Reynolds number $R = 10^4$ and $\alpha = 0.179$ (cf. Mack [5]). The results are presented in Fig. 3, where the relative errors of $c(z_s)$ obtained with the conventional and with the corrected starting data are compared. The reference eigenvalues c_∞ were taken as $c(y_s = 14)$: $c_\infty^{(1)} = 0.32500688 - i\,0.03246144$, $c_\infty^{(5)} = 0.26864673 - i\,0.09745747$, $c_\infty^{(9)} = 0.41657957 - i\,0.19608952$, which differ slightly from Mack's values possibly because of a different interpolation scheme for the velocity profile. To give an idea of the magnitude of the corrections φ_1, the quantities $\Delta_\alpha^m = |D^m \varphi_1^{(-\alpha)}/D^m \varphi_0^{(-\alpha)}|\,(y = y_s)$ and $\Delta_\beta^0 = |\varphi_1^{(-\beta)}/\varphi_0^{(-\beta)}|(y = y_s)$ arising in the calculation of Fig. 3 are plotted in

Fig. 4 as a function of z_s^2 for mode 1; neither the corrections for the higher derivatives Δ_β^m nor the corrections for the higher modes 5 and 9 are presented as they follow closely Δ_β^0 (confirming estimate (18)) and the corresponding corrections for mode 1, respectively.

From Fig. 3 it is concluded that the error incurred with the conventional starting data depends analytically on z_s and behaves essentially like $\exp(-z_s^2)$ whereas the improved starting data yield a reduction of the error by another factor $\exp(-z_s^2)$. The calculations for the higher modes show the same behaviour. As long as an orthogonalization scheme [7] is used which preserves the analytic dependence of the iteration function $\phi(0)$ on c, no basic difficulty is encountered at smaller z_s. In the neighborhood of eigenvalues $c(z_s)$, $\phi(0)$ is approximately given by the first term of its Taylor expansion proportional to $c - c(z_s)$ with the proportionality constant increasing drastically with the mode number. Thus the calculation of higher eigenvalues is not a problem of the iteration scheme—even the simplest Newton procedure will yield a rapid convergence—but a problem of resolution. This is where the

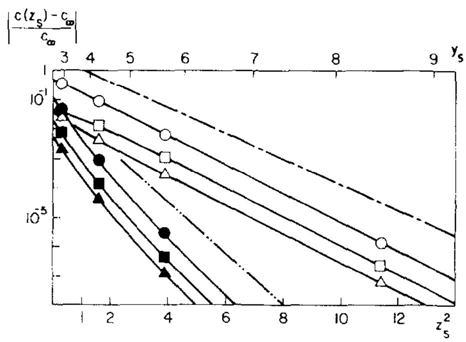

FIG. 3. Relative error of the eigenvalue c versus z_s with starting data $D^m\varphi_0$ (open symbols) and with corrected starting data $D^m(\varphi_0 + \varphi_1)$ (solid symbols). Temporal calculation with $R = 10^4$ and $\alpha = 0.179$ for mode 1 (○), mode 5 (□) and mode 9 (△). – · –, – ·· –: slopes of $\exp(-z_s^2)$ and $\exp(-2z_s^2)$, respectively.

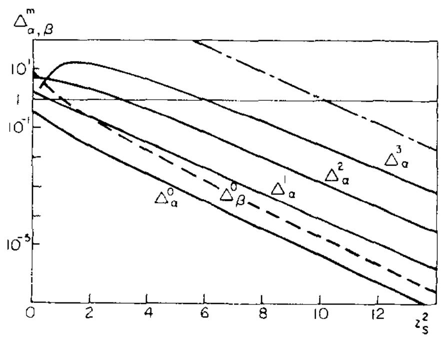

FIG. 4. Relative magnitudes $\Delta_\alpha^m = |D^m\varphi_1^{(-\alpha)}/D^m\varphi_0^{(-\alpha)}|$(—) and $\Delta_\beta^0 = |\varphi_1^{(-\beta)}/\varphi_0^{(-\beta)}|$(– – –) of the corrections to the "free-stream" solutions versus z_s for the mode 1 calculation of Fig. 3.

corrected starting data provide the possibility of reducing the number of integration steps and/or the step size to increase the number of significant digits of $\phi(0)$. If on the other hand a nonanalytic orthogonalization scheme is used as, for instance, the Gram–Schmidt procedure in [5], it appears that the surface $\phi(0)$ becomes more and more distorted for smaller starting locations z_s to a point where an automatic iteration scheme no longer converges.

To conclude this section, we discuss also the case where U is given asymptotically by a series of the type

$$U_{as} = \lambda + \sum_{n=1}^{\infty} u_n \exp(n\vartheta y). \tag{19}$$

This situation occurs on the zero-velocity side of a Blasius mixing layer with $\Lambda = 0$ (cf. expression (10) for F_{as}) or if, for instance, a $\tanh(y)$ profile for U is assumed. In this case the correction of the "free-stream" starting data is even more important, as an eigenvalue from a numerical calculation starting at y_s will approach its exact value only like $\exp(\vartheta y_s)$. If (19) is inserted into the O–S equation (12) the asymptotic expansion for the solution ϕ is easily obtained as

$$\phi \propto \varphi_0 \left\{ \sum_{n=0}^{\infty} d_n \exp(n\vartheta y) \right\} \tag{20}$$

with φ_0 given by (13a), (14) and

$$\begin{aligned} d_0 &= 1, \\ d_n &= \frac{i\alpha R}{n\vartheta} \left\{ \sum_{k=1}^{n} u_k d_{n-k} [D^2\varphi_0 + 2\vartheta(n-k) D\varphi_0 \right. \\ &\quad \left. + (n^2\vartheta^2 - 2nk\vartheta^2 - \alpha^2)\varphi_0] \right\} \left\{ 4D^3\varphi_0 + 6n\vartheta D^2\varphi_0 \right. \\ &\quad \left. + [4n^2\vartheta^2 - 2(\alpha^2+\beta^2)] D\varphi_0 + n\vartheta[n^2\vartheta^2 - (\alpha^2+\beta^2)]\varphi_0 \right\}^{-1}. \end{aligned}$$

This last recurrence relation for the coefficients d_n, in which φ_0 cancels out, is easily programmed on a computer. Thus, the starting data at any $|y_s| > 0$ can be improved in this case to any desired accuracy with no significant increase in computing time.

4. Conclusions

It has been shown that in the case of an infinite domain the accuracy of the results obtained from a numerical integration of the O–S equation can be greatly improved by correcting the starting data for the finite computational domain using the asymptotic corrections of the "free-stream" solutions. The success of incorporating these corrections into a shooting algorithm for the eigenvalue is exemplified by three Blasius boundary-layer calculations.

Practically, the results can be exploited in two different ways: to improve the eigenvalue with a given starting point y_s or to reduce the computation by starting the numerical calculation closer to the shear layer.

From the theoretical standpoint another view of the results is also useful: let us consider the boundary-value problem in the finite computational domain and require, as a boundary condition at y_s, that the solution be a linear combination of the two asymptotically damped "free-stream" solutions only, which has been the usual approach so far. Now we may ask which combination of solutions has to be taken in an exact integration starting at infinity to yield the above boundary condition at y_s. Using results (17) or (20) it is easy to show that this exact combination has to contain also the two asymptotically growing solutions with amplitudes which are of the order of the correction to the asymptotically constant mean velocity evaluated at y_s. In the Blasius case with $\lambda > 0$ they are therefore $O(\exp(-z_s^2))$, and in the case of the profile (19), $O(\exp(\vartheta y_s))$.

Another point of interest is the factor $(\lambda - c)^{-1}$ in correction (17) which can become large close to the continuous spectrum (cf. Grosch and Salwen, [3]). If computations are undertaken in the vicinity of the endpoint of the continuous spectrum $c = \lambda - i\alpha/R$, as was done by Antar and Benek [1], then $(\lambda - c)^{-1}$ yields a factor which in the worst case is large of the order $O(R)$. This is not surprising, as in the vicinity of the continuous spectrum the asymptotic behaviour is only slowly approached by the eigenfunctions.

References

1. B. N. Antar and J. A. Benek, *Phys. Fluids* **21** (1978), 183–189.
2. H. Blasius, *Z. Math. Phys.* **56** (1908), 1–37 (cf. also NACA TM 1256).
3. C. E. Grosch and H. S. Salwen, *J. Fluid Mech.* **87** (1978), 33–54.
4. R. C. Lock, *Quart. J. Mech. Appl. Math.* **4** (1951), 42–63.
5. L. M. Mack, *J. Fluid Mech.* **73** (1976), 497–520.
6. P. A. Monkewitz and M. A. Monkewitz, *Bull. Amer. Phys. Soc.* **23/8** (1978), 994.
7. M. A. Monkewitz, *Z. Angew. Math. Phys.* **29** (1978), 861–870.
8. B. S. Ng and W. H. Reid, *J. Comput. Phys.*, in press.
9. H. Schlichting, "Boundary-Layer Theory," 6th ed., McGraw–Hill, New York, 1968.